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LETTER TO THE EDITOR

Supersymmetric transformations and Hamiltonians generated by the Marchenko equations

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Abstract. Three different isospectral Hamiltonians have been generated by eliminating the ground state of a given Hamiltonian using procedures based on the Marchenko equations for left- and right-incident waves and the standard model of supersymmetric quantum mechanics. It is shown that each of the two procedures based on the Marchenko equation is equivalent to the application of two appropriately chosen supersymmetric transformations.

Let

$$H_1 = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_1(x) \qquad -\infty \leqslant x \leqslant \infty \tag{1}$$

be a Hamiltonian with bound states $\psi_1(E_i)$ at energies E_i, $i = 1, 2, \ldots, n$. Three different procedures for eliminating the ground state of H_1 and generating a new Hamiltonian with bound states at energies E_i, $i = 2, 3, \ldots, n$, have been discussed in the literature. The first procedure based on the Gel'fand–Levitan integral equation (Gel'fand and Levitan 1951) was discussed by Abraham and Moses (1980) and will hereafter be referred to as AM. For the spatial domain $-\infty \leqslant x \leqslant \infty$ the procedure used by AM is equivalent to using the Marchenko equation (Marchenko 1955) for waves incident from the left (Newton 1980). The second procedure (Pursey 1986) is based on the Marchenko equation for waves incident from the right and will hereafter be referred to as M. The third procedure is based on a standard model of supersymmetric quantum mechanics (Witten 1981) and finds a partner Hamiltonian to H_1 by using an appropriate factorisation of H_1 (Andrianov *et al* 1984, Sukumar 1985a, b). The third procedure has been identified as being equivalent to the use of the Darboux transformation for second-order differential equations (Darboux 1882) and will be referred to as D. The three new Hamiltonians generated by the three procedures have the same spectrum of eigenvalues but in general have different reflection and transmission coefficients for positive energies (Pursey 1986, Luban and Pursey 1986). It is the purpose of this letter to show that the procedures used by AM and M are each equivalent to a different two-step procedure based on two appropriately chosen D transformations.

It has been shown that the ground-state eigenfunction $\psi_1(E_1)$ of H_1 may be used to factorise H_1 in the form

$$H_1 = A_1^+(E_1)A_1^-(E_1) + E_1 \tag{2}$$

$$A_1^{\pm}(E_1) = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} + \left\{\frac{\mathrm{d}}{\mathrm{d}x}\ln\psi_1(E_1)\right\}\right). \tag{3}$$

$\psi_1(E_1)$ will be assumed to be normalised to unity throughout this paper. The D transformation generates the partner Hamiltonian

$$H_2 = A_1^-(E_1)A_1^+(E_1) + E_1 = H_1 - (\mathrm{d}^2/\mathrm{d}x^2) \ln \psi_1(E_1) \tag{4}$$

whose spectrum is identical to that of H_1 except for the missing ground-state energy E_1. The eigenstates of H_2 are given by

$$\psi_2(E_i) \sim A_1^-(E_1)\psi_1(E_i) \qquad i = 2, 3, \ldots, n. \tag{5}$$

Even though E_1 is not an eigenenergy of H_2, a formal solution of the Schrödinger equation for H_2 at energy E_1 may be given as

$$\varphi_2(E_1) \sim [\psi_1(E_1)]^{-1}. \tag{6}$$

A second linearly independent solution given by

$$\tilde{\varphi}_2(E_1) = \varphi_2(E_1) \int^x [\varphi_2(E_1)]^{-2}\,\mathrm{d}y \tag{7}$$

may be used to give the general solution

$$\psi_2(E_1, \lambda) = [\psi_1(E_1)]^{-1}\left(1 + \lambda \int_{-\infty}^{x} \psi_1^2(E_1)\,\mathrm{d}y\right). \tag{8}$$

$\psi_2(E_1, \lambda)$ may then be used to factorise H_2 in the new form

$$H_2 = A_2^+(E_1, \lambda)A_2^-(E_1, \lambda) + E_1 \tag{9}$$

$$A_2^{\pm}(E_1, \lambda) = \frac{1}{\sqrt{2}}\left(\pm\frac{\mathrm{d}}{\mathrm{d}x} + \left\{\frac{\mathrm{d}}{\mathrm{d}x} \ln \psi_2(E_1, \lambda)\right\}\right). \tag{10}$$

This new factorisation allows the identification of a new supersymmetric partner to H_2 by the application of a second D transformation and is given by

$$\begin{aligned} H_3(\lambda) &= A_2^-(E_1, \lambda)A_2^+(E_1, \lambda) + E_1 \\ &= H_2 - (\mathrm{d}^2/\mathrm{d}x^2) \ln \psi_2(E_1, \lambda) \\ &= H_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln\left(1 + \lambda \int_{-\infty}^{x} \psi_1^2(E_1)\,\mathrm{d}y\right). \end{aligned} \tag{11}$$

The eigenfunctions of H_3 for energies E_i, $i = 2, 3, \ldots, n$, are given by

$$\psi_3(E_i, \lambda) \sim A_2^-(E_1, \lambda)\psi_2(E_i) \qquad i = 2, 3, \ldots, n. \tag{12}$$

Furthermore,

$$\psi_3(E_1, \lambda) \sim [\psi_2(E_1, \lambda)]^{-1}. \tag{13}$$

Equations (12), (10), (5) and (3) may be used to generate the eigenfunctions of H_3 for energies E_i, $i \neq 1$, in the form

$$\begin{aligned} \psi_3(E_i, \lambda) = \psi_1(E_i) - \frac{1}{2(E_i - E_1)} \frac{\lambda\psi_1(E_1)}{[1 + \lambda \int_{-\infty}^{x} \psi_1^2(E_1)\,\mathrm{d}y]} \\ \times \left\{\psi_1(E_1)\frac{\mathrm{d}}{\mathrm{d}x}\psi_1(E_i) - \psi_1(E_i)\frac{\mathrm{d}}{\mathrm{d}x}\psi_1(E_1)\right\} \qquad i = 2, 3, \ldots, n. \end{aligned} \tag{14}$$

The relation between the spectra of H_2 and the one-parameter family of Hamiltonians $H_3(\lambda)$ depends on the choice of value for the parameter λ.

(i) $-\infty < \lambda < -1$. $\psi_2(E_1, \lambda)$ has a node at a finite value of x and the corresponding H_3 will be singular at that value of x. Hamiltonians with such singularities may be rejected on physical grounds.

(ii) $-1 < \lambda < \infty$. $\psi_2(E_1, \lambda)$ is nodeless and $\psi_3(E_1, \lambda)$ given by (13) is normalisable. For these values of λ, H_3 has a true bound state at energy E_1 with $\psi_3(E_1, \lambda)$ as the ground-state eigenfunction. Hence H_3 and H_1 have identical spectra. It is easy to show from (8), (13) and (14) that the ground state of H_3 is renormalised while the other eigenstates are not. For $\lambda = 0$, $H_3 = H_1$. For other values of λ, H_3 is a member of the phase-equivalent family for H_1 (Sukumar 1985b).

(iii) $\lambda = -1$. Equation (8) may be used to give

$$\tilde{\psi}_2(E_1) \equiv \psi_2(E_1, -1) \sim [\psi_1(E_1)]^{-1} \int_x^{\infty} \psi_1^2(E_1)\, \mathrm{d}y \tag{15}$$

$$\tilde{H}_3 \equiv H_3(E_1, -1) = H_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \int_{\infty}^{x} \psi_1^2(E_1)\, \mathrm{d}y. \tag{16}$$

Equations (13) and (15) show that $\psi_3(E_1, -1)$ is not a normalisable solution. Hence E_1 is not an eigenenergy of $\tilde{H}_3$. $\tilde{H}_3$ and H_2 have identical spectra. It is then possible to conclude that $\tilde{H}_3$ has a spectrum identical to that of H_1 except for missing the ground state of H_1. The eigenfunctions of $\tilde{H}_3$ for the eigenenergies E_i, $i = 2, 3, \ldots, n$, are given by (14) with $\lambda = -1$. $\tilde{H}_3$ is identical to the Hamiltonian generated by AM (see also Chaturvedi and Ragunathan 1986).

(iv) $\lambda = \infty$. Equation (8) may be used to give

$$\tilde{\tilde{\psi}}_2(E_1) \equiv \psi_2(E_1, \infty) \sim [\psi_1(E_1)]^{-1} \int_{-\infty}^{x} \psi_1^2(E_1)\, \mathrm{d}y \tag{17}$$

$$\tilde{\tilde{H}}_3 \equiv H_3(E_1, \infty) = H_1 - \frac{\mathrm{d}^2}{\mathrm{d}x^2} \ln \int_{-\infty}^{x} \psi_1^2(E_1)\, \mathrm{d}y. \tag{18}$$

Equations (13) and (17) show that $\psi_3(E_1, \infty)$ is not a normalisable solution and E_1 is not an eigenenergy of $\tilde{\tilde{H}}_3$. Therefore, $\tilde{\tilde{H}}_3$, H_2 and $\tilde{H}_3$ have identical spectra. The eigenfunctions of $\tilde{\tilde{H}}_3$ for the eigenenergies E_i, $i = 2, 3, \ldots, n$, are given by (14) with $\lambda = \infty$. $\tilde{\tilde{H}}_3$ is identical to the Hamiltonian generated by M.

We have shown that the procedures for eliminating the ground state based on the Marchenko equations for left and right incidences are each equivalent to the application of two Darboux transformations. It is easy to show that a parallel analysis may be carried out for the case of the addition of a new ground state and that the application of two appropriately chosen D transformations produces the same potentials as those generated by the AM and M procedures for the corresponding case.

The above discussion for the x space $(-\infty \leqslant x \leqslant \infty)$ may now be compared with the discussion for the r space $(0 \leqslant r \leqslant \infty)$ given in Sukumar (1985c), where four different types of possible transformations of the radial analogue of H_1 were identified. It will be assumed from now on that $H_1(r)$ has centrifugal terms corresponding to angular momentum l. It was shown that a transformation identified as T_1 eliminates the ground state of H_1 and changes the behaviour of the eigenfunctions in the limit $r \to 0$ from r^l to r^{l+1}. A transformation identified as T_2 was used to add a state below the ground state of H_1 and alter the eigenfunction behaviour as $r \to 0$ from r^l to r^{l-1}. It was shown that a transformation T_3 maintains the spectrum of H_1 but alters the limiting value of ψ as $r \to 0$ from r^l to r^{l+1}, while a transformation T_4 maintains the spectrum of H_1 and alters the behaviour of the eigenfunctions as $r \to 0$ from r^l to r^{l-1}. Each of the four transformations T_1-T_4 changes the angular momentum state as identified by noting the behaviour near $r=0$. It must be emphasised that for all potentials other than the Coulomb potential each of the four transformations leaves unaltered the angular momentum state as identified by the wavefunction behaviour as $r \to \infty$. Therefore, no simple identification of the centrifugal part of the new potentials generated by a single application of one of the transformations T_1-T_4 is possible. The radial analogue of the AM procedure for eliminating the ground state has been shown (Sukumar 1985c) to be equivalent to the application of T_1 followed by T_4. The new Hamiltonian arising after T_1 and T_4 is the radial analogue of $\tilde{H}_3$ (equation (16)). It is easy to show that the new eigenfunctions after the two transformations have behaviour as $r \to 0$ and $r \to \infty$ unaltered from that for the eigenfunctions of $H_1(r)$, i.e. the centrifugal part of the $\tilde{H}_3(r)$ is easily identified to be the same as that for H_1.

The radial analogue of the M procedure is generated by the application of T_1 followed by T_3. The resulting new Hamiltonian can be identified as the radial analogue of $\tilde{\tilde{H}}_3$ (equation (18)). It is easy to show that, after the application of T_1 and T_3, the new eigenfunctions have their behaviour as $r \to 0$ changed from r^l to r^{l+2} while the angular momentum state as reflected by the $r \to \infty$ limit of the eigenfunctions is unaltered from that for the eigenfunctions of $H_1(r)$, i.e. the potential term in the radial analogue of $\tilde{\tilde{H}}_3$ would have centrifugal-like terms for short-range values of r even after the centrifugal term $l(l+1)/r^2$ is subtracted out. Therefore, no simple identification of the centrifugal part of the new potential generated by the radial analogue of the M procedure is possible.

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